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Spontaneous symmetry breaking in $U(N)$ invariant ensembles with a soft confinement potential

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A solution is provided to the problem of finding the probability distribution of elements of a random matrix in terms of the distribution of eigenvalues and eigenvectors. It is then proved that completely isotropic eigenvectors can become localized when the eigenvalues increase exponentially. This general result confirms the prediction of a spontaneous breaking of the unitary transformation, *U(N)*, invariance of random matrix ensembles, in the limit of extremely soft confinement. An algorithm is implemented to generate eigenvectors with broken symmetry. The theory is then verified numerically.

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The ensembles of random matrices [1] introduced in the 1950's by Wigner have found, especially in the last two decades, broad application in many areas of physics [2]. In their standard version, which constitutes what is usually called random matrix theory (RMT), the matrix elements are Gaussian distributed and the joint probability of eigenvectors and eigenvalues factorize, with the latter obeying the Wigner-Dyson (WD) statistics while the former behave as isotropic eigenfunctions in Hilbert space. The main characteristic of WD is the strong correlation caused by the presence of repulsion between levels, materialized in the Wigner distribution for the level spacings. WD statistics are expected to be observed in chaotic systems of few or many degrees of freedom [3] and, also, in disordered quantum systems in their metallic phase [4] .

Contrary to WD, the superposition of an uncorrelated sequence of levels is known to follow Poisson statistics [1]. In this case, since there is no level repulsion to prevent, they can cluster, and the spacing distribution has its maximum at zero separation. The lack of correlation in the eigenfunctions is expected to correspond to localized states that don't ''talk'' to each other and show a multifractal behavior. These kinds of statistics are expected to be found in regular systems and in the insulator phase of quantum disordered systems [5]. A statistical model with these properties would be the ensemble of large diagonal matrices whose elements are Gaussian distributed.

In recent applications of statistical models of spectra, the crossover from the WD to Poisson statistics has become a major issue. Since RMT ensembles are constructed by requiring invariance with respect to *U(N)*, i.e., unitary transformations, a direct way to generate ensembles to interpolate RMT and Poisson statistics is to impose a condition that breaks, by construction, this invariance. This is done by introducing a preferred basis, which means, in practical terms, going from a complete matrix space to a subspace of sparse diagonal or near-diagonal, matrices. This procedure has been followed, for example, in the construction of deformed ensembles [6], of random band matrix ensembles [7], and also of the ensembles of Refs. [8] and [9].

Another approach, quite different in principle, consists in trying to obtain the transition without breaking the rotational invariance of RMT. In this case, the eigenvalue distribution has the usual RMT form

$$P(x_1,x_2,\ldots,x_N)=C_N\exp\left[-\sum_{k=1}^{N}V(x_k)\right]\prod_{j>i}|x_j-x_i|^{\gamma}, \tag{1}$$

where the symmetry parameter γ has the values $\gamma=1$, 2, and 4 for the orthogonal, unitary, and symplectic cases, respectively, and C_N is a normalization constant. In Eq. (1), the stringent parabolic choice for $V(x)$ of RMT is, then, replaced, by a potential that asymptotically provides only a soft confinement to the eigenvalues [10,11]. In Ref. [11], it was argued that one should expect a departure from the WD statistics for potentials that lead to an ''incompressible phase'' in which the level density becomes independent of the total number of eigenvalues N, in the limit in which $N\to\infty$. As is well known, the semicircle law that gives the level density for the Gaussian ensembles grows as $\sqrt{N}$ for large N.

Numerical simulations and, also, analytical derivation [11,12] have shown that a nonstandard behavior indeed occurs for a logarithmic confinement of the form $V(x)=(1/\beta)\ln^2|x|$ when the parameter β is made large. In this case, the average density, calculated with Dyson's mean-field approximation, behaves as

$$\rho(x)\propto\frac{1}{|x|}. \tag{2}$$

In this limit of large β, however, the spacings between levels are not distributed according to Poisson but, instead, follow a new Poisson-like distribution which is the same for the three symmetries classes, i.e., the orthogonal, the unitary, and the simplectic [11]. Some new properties are also exhibited by the eigenvalues, such as lack of translational invariance and the presence of a long-range correlation that prohibits two eigenvalues from being located at symmetric distances from the origin. These features and, also, the idea that there must be a connection between eigenvalue and eigenvector distributions, have suggested that there should be a spontaneous breaking of the rotational symmetry [13].

The purpose of this paper is to show analytically how this spontaneous symmetry breaking occurs and to present an algorithm that generates numerically eigenfunctions with broken symmetry. This will be done by first constructing a solution of what one may call the inverse problem since, contrary to the usual situation, here the distributions of eigenvalues and eigenvectors are given and from them, the probability distribution of the matrix itself is to be derived. In other words, a meaning will be given to the formal equation

$$P(\mathbf{M})=K_N \exp\left[-\frac{1}{\beta}\,\mathrm{tr}(\ln^2|\mathbf{M}|)\right], \tag{3}$$

and it will then shown that in the limit $\beta\to\infty$, the probabilty $P(\mathbf{M})$ becomes singular, yielding, as a consequence, the localization of the eigenvectors. To be specific and to simplify the derivation, only the orthogonal case, $\gamma=1$, will be considered.

The *U(N)* invariance of the ensemble implies eigenvectors completely isotropic in the *N*-dimensional space with the restrictions imposed by the normalization and the orthogonality. If we look at a particular eigenvector, say, $|x_k\rangle$ of the eigenvalue x_k, and forget all others, the distribution of its components, $C_i^k=\langle i|x_k\rangle$ with respect to a set of base vectors $|i\rangle$, can be written as

$$P(C_1^k,C_2^k,\ldots,C_N^k)=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\delta\left[1-\sum_{i=1}^{N}(C_i^k)^2\right]. \tag{4}$$

From this equation, the distribution

$$P(y)=\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}y^{-1/2}(1-y)^{(N-3)/2} \tag{5}$$

for the strength $y=C^2$ follows, derived by integration into all components but one, which reduces to the well known signature of RMT, the Porter-Thomas law [14], for large *N*.

Considering now the eigenvalue distribution, Eq. (2) suggest the substitution [11]

$$x=\mathrm{sgn}(\xi)\exp(\beta|\xi|) \tag{6}$$

that maps the variables onto the domain $-(N/2)\leqslant\xi\leqslant(N/2)$. In terms of this new variable, the eigenvalue joint probability distribution becomes

$$P(\xi_1,\ldots,\xi_N)=C_N\exp\left[-\beta\sum_{k=1}^{N}(\xi_k^2-|\xi_k|)\right]\times\prod_{j>i}|\mathrm{sgn}(\xi_j)\exp(\beta|\xi_j|)-\mathrm{sgn}(\xi_i)\exp(\beta|\xi_i|)|.$$

Assuming the eigenvalues are ordered as

$$|\xi_1|>|\xi_2|>\cdots>|\xi_N|, \tag{7}$$

the distribution can be rewritten as

$$P(\xi_1,\xi_2,\ldots,\xi_N)=C_N\exp\left[-\beta\sum_{k=1}^{N}(\xi_k^2-k|\xi_k|)\right] \tag{8}$$

or, finally [11],

$$P(\xi_1,\xi_2,\ldots,\xi_N)=\prod_{k=1}^{N}\sqrt{\frac{\beta}{4\pi}}\exp\left[-\beta\left(|\xi_k|-\frac{k}{2}\right)^2\right]. \tag{9}$$

The scaled eigenvalues therefore place themselves around the points $\pm k/2$ for $k=1,2,\ldots,N$, with the restriction that symmetric sites are not occupied simultaneously, which implies in the long-range correlation cited above. This result has been confirmed by numerical simulations [11].

Matrix elements are connected to eigenvalue and eigenvector components through the relation

$$M_{ij}=\sum_{k=1}^{N}x_kC_i^kC_j^k. \tag{10}$$

Combining Eqs. (10) and (4), we can write the probability of a given diagonal element M_{ii} as

$$P(M_{ii})=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\int d^Nx\,d^NC^k\,\delta\left[M_{ii}-\sum_{k=1}^{N}x_k(C_i^k)^2\right]\times\delta\left[1-\sum_{i=1}^{N}(C_i^k)^2\right]P(x_1,\ldots,x_N). \tag{11}$$

This equation establishes the general connection between the probability distribution of diagonal matrix elements and the probability distributions of the eigenvalues and eigenvectors when they are isotropically distributed.

To show how the breaking of the rotational invariance occurs, it is instructive to begin by considering an ensemble of 2×2 matrices. In this case, the components of the two eigenvectors can be expressed as functions of the angle θ, which gives the direction of the eigenvector in the plane of the base. In terms of θ, Eq. (10) becomes

$$\begin{aligned}M_{11}&=x_1\cos^2\theta+x_2\sin^2\theta,\\M_{12}&=(x_1-x_2)\sin\theta\cos\theta,\\M_{22}&=x_1\sin^2\theta+x_2\cos^2\theta.\end{aligned} \tag{12}$$

Keeping the two eigenvalues at fixed values, x_1,x_2, the probability $P(M_{11},x_1,x_2)$ that the first diagonal matrix element has a value M_{11} is given by

$$P(M_{11},x_1,x_2)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\delta(M_{11}-x_1\cos^2\theta-x_2\sin^2\theta) \tag{13}$$

or, after the integration,

$$P(M_{11},x_1,x_2)=\frac{1}{\pi}\frac{1}{\sqrt{(x_1-M_{11})(M_{11}-x_2)}}. \tag{14}$$

Introducing the scaled variables ξ_1, ξ_2, and ξ with $M_{11}=\mathrm{sgn}(\xi)\exp(\beta\xi)$, we have

$$P(\xi,\xi_1,\xi_2)=\frac{1}{\pi}\frac{\exp[\beta(|\xi|-|\xi_1|)]}{\sqrt{\{\text{sgn}(\xi_1)-\text{sgn}(\xi)\exp[-\beta(|\xi_1|-|\xi|)]\}}}\frac{1}{\sqrt{\{\text{sgn}(\xi)\exp[-\beta(|\xi_1|-|\xi|)]-\text{sgn}(\xi_2)\exp[-\beta(|\xi|-|\xi_2|)]\}}},\tag{15}$$

where it was assumed that $|x_1|>|x_2|$. Taking the limit $\beta\to\infty$, it follows that

$$P(M_{11})\to\begin{cases}0 & \text{if} \quad M_{11}\neq x_1\\ \infty & \text{if} \quad M_{11}=x_1,\end{cases}\tag{16}$$

which means that for large β the probability of finding a value of M_{11} with a value other than x_1 vanishes. Replacing this value in the first Eq. (12) we obtain

$$x_1\sin^2\theta=x_2\sin^2\theta,$$

which can only be satisfied if $\theta=0,\pi$. For these two values, the matrix is diagonal and the eigenvectors are necessarily of the form $C_i^k=\pm\delta_{ik}$.

On the other hand, if the off-diagonal element is considered, we find the probability distribution

$$P(M_{12},x_1,x_2)=\frac{1}{\pi}\frac{4}{\sqrt{(x_1-x_2)^2-4M_{12}^2}},$$

in which case, the angle has the values $\theta=\pm\pi/4$ or $\pm3\pi/4$.

Considering now ensembles with matrices of sizes greater than 2, the integrals in the components in Eq. (11) can be performed by introducing Fourier representations for the δ functions. Again keeping the eigenvalues fixed, we have the probability distribution

$$P(M_{ii},x)=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\frac{\pi^{N/2}}{(2\pi)^2}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1dk_2\exp(ik_1M_{ii}+ik_2)}{\sqrt{\Pi_{k=1}^{k=N}[i(k_1x_k+k_2)]}},\tag{17}$$

making the substitution $k_2=k_1k$,

$$P(M_{ii},x)=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\frac{\pi^{N/2}}{(2\pi)^2}\int_{-\infty}^{\infty}dk_1k_1^{-[(N/2)-1]}\times\text{sgn}(k_1)\int_{-\infty}^{\infty}\frac{dk_2\exp[ik_1(M_{ii}+k_2)]}{\sqrt{\Pi_{k=1}^{k=N}[i(x_k+k_2)]}}.$$

Reintroducing the integrations in the C variables, the integration in k_2 can be performed, giving a δ function that allows the k_1 integration also to be performed, and one arrives at the expression

$$P(M_{ii},x)=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\int\frac{d^NC}{(\Sigma_{i=1}^N C_i^2)^{(N/2)-1}}\times\exp\left[-i\sum_{k=1}^N(M_{ii}-x_k)C_i^2\right].$$

Using the Fourier representation [15]

$$\left(\sum_{i=1}^N C_i^2\right)^{-(N/2)+1}=\frac{1}{2\Gamma\left(\frac{N}{2}-1\right)\sin\frac{N\pi}{4}}\int dk_1|k_1|^{N/2}\quad\text{for }-2\times\exp\left(ik_1\sum_{i=1}^N C_i^2\right),$$

the integrations in the C variables are performed and we find

$$P(M_{ii},x)=\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\frac{1}{2\Gamma\left(\frac{N}{2}-1\right)\sin\frac{N\pi}{4}}\times\mathcal{P}\int_{-\infty}^{\infty}\frac{|k_1|^{(N/2)-2}dk_1}{\sqrt{\Pi_{k=1}^{k=N}[i(x_k-M_{ii}+k_1)]}},$$

where $\mathcal{P}$ stands for the Cauchy principal value. Since this equation was derived without any assumption about the eigenvalues, it gives the general dependence of diagonal matrix element distribution for isotropic eigenvectors.

To see how the limit $\beta\to\infty$ of our ensemble, i.e., eigenvalues distributed according to Eq. (9), introduces localization in the above expression, we start by making the substitution $k_1=|x_1|k_2$, x_1, denoting the largest eigenvalue $P(M_{ii},x)$

$$\sim\frac{1}{|x_1|}\mathcal{P}\int_{-\infty}^{\infty}\frac{|k_1|^{(N/2)-2}dk_1}{\sqrt{|(x_1-M_{ii})/|x_1|+k_1||k_1-M_{ii}/|x_1||^{N-1}}}.$$

We immediately see that for $M_{ii}\neq x_1$, $P(M_{ii},x)$ vanishes when $\beta\to\infty$. On the other hand, if $M_{ii}=x_1$ then the above integral can be asymptotically written as

$$P(M_{ii},x)\sim\frac{1}{|x_1|}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk_1}{k_1^2\sqrt{|1-1/k_1|^{N-1}}},$$

which diverges because of the singularity at $k_1=1$. Now, making $M_{ii}=x_1$, Eq. (10) becomes

$$\sum_{k=2}^N(x_1-x_k)(C_1^k)^2=0.$$

Dividing it by x_1, and taking the limit $\beta\to\infty$, we find that $C_1^k=0$ for $k\neq1$ so $C_1^k=\pm\delta_{1k}$.

Considering now the probability distribution of other matrix elements, the eigenvalue x_1 will be absent. Therefore, assuming x_2 to be the second largest eigenvalue, we are going to find that the next matrix element, say, M_{22}, has to be equal to it and, as before, $C_2^k=\pm\delta_{2k}$. Proceeding with this calculation, we are going to conclude what we want to prove, namely, that in the limit of large β, we have an ensemble of diagonal matrices with completely localized eigenvectors of the form $C_i^k=\pm\delta_{ik}$.

In principle, the same procedure can be used to discuss what happens with the distributions of the off-diagonal elements. But it is more convenient to introduce here a different approach. First we observe that the probability distribution of the matrix elements concentrates at the extremal values of the matrix elements, as functions of the eigenvectors keeping the eigenvalues fixed. To see that this is indeed the case in the above results for the diagonal elements, let us maximize Eq. (10) with $i=j$. Introducing a Lagrange multiplier, λ, to take into account the normalization of the eigenvector, we get, by differentiation, the set of N equations

$$(x_k-\lambda)C_k^i=0, \quad k=1,N,$$

whose immediate solutions are $\lambda=x_k$ and $C_k^i=\pm\delta_{ik}$ with $k=1,N$, in agreement with prior results.

Considering now off-diagonal matrix elements, we need three Lagrange multipliers to incorporate the two normalizations and the orthogonality condition. Differentiation, then, leads to $2N$ equations

$$(x_k-\lambda_1)C_k^j-\lambda_2 C_k^i=0,$$

$$(x_k-\lambda_1)C_k^i+\lambda_3 C_k^j=0, \quad k=1,N$$

that have the solutions $\lambda_1=(x_k+x_l)/2$, $\lambda_2=\lambda_3=(x_k-x_l)/4$, and $C_{k,l}^i=C_{k,l}^j=\pm 1/\sqrt{2}$, where (k,l) is a given pair of indices and the other components vanish.

Therefore, when the symmetry is broken, one should expect the distribution of the strength, $y=C^2$, to be concentrated at the points $y=0,\frac{1}{2},1$. To verify this prediction, a numerical simulation was performed whose results are shown in Fig. 1. In the calculation, Gaussian orthogonal ensembles (GOEs), i.e., isotropic eigenvectors, were generated and matrix elements were obtained using Eq. (10) with eigenvalues given by $x_k=\exp(\beta k/2)$. To induce the symmetry breaking, a small perturbation is added by multiplying the off-diagonal terms by the factor 0.99999. The matrix is then diagonalized. That is how the two histograms of Fig. 1 were obtained, while the solid line corresponds to GOE, i.e., Eq. (5). As can be seen in the figure, the perturbation has no effect for small β. However, for the value, $\beta=40$, the distribution changes dramatically, with a substantial increase in the maximum at the origin and the appearance of peaks at $y=\frac{1}{2}$ and $y=1$, as predicted.

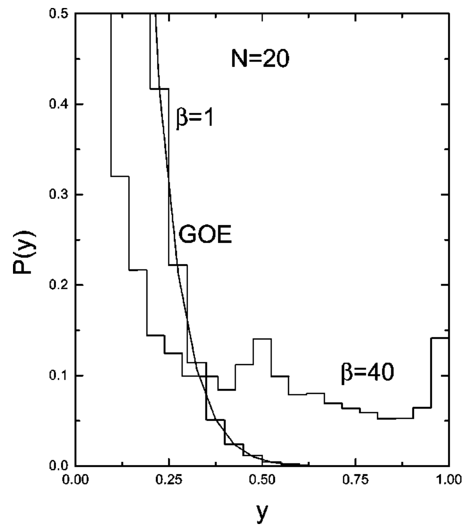

FIG. 1. Histograms of $y=C^2$ for the two indicated values of the parameter β. The solid line is the GOE result, Eq. (5). The calculations correspond to matrices of dimension $N=20$.

In conclusion, it has been analytically proven, and numerically verified, that completely isotropic eigenvectors can become localized if the eigenvalues increase exponentially. This result is general and confirms the prediction [13] that the *U(N)* invariance of a random matrix ensemble with a soft confinement potential, $V(x)=(1/\beta)\ln^2|x|$, is broken in the limit of large values of the parameter β. Rigorously, the result is valid only in the limit $\beta\rightarrow\infty$. An investigation into what happens for finite or even small β is lacking. The method, however, especially its implementation as an algorithm to generate eigenvectors, opens the possibilty of a more complete investigation of the critical point of disordered quantum systems, using soft confinement ensembles. As a last remark, although for simplicity only the orthogonal case was considered, it should be straightforward to extend the analysis to the unitary and simplectic ensembles.

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